Subject	class	Paper	Study materials on	Resource Person
Mathematics	D-I(H)	I	Transformation of Eqn	Dr. S. Ahmad, Associate Professor

Ex. If α, β, γ be the roots of the equation $x^3+qx+r=0$. find the equation whose roots are $\frac{\beta}{\gamma}+\frac{\gamma}{\beta}$, $\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}$, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

Soln Here $\alpha+\beta+\gamma=0$

$\beta+\gamma=-\alpha$

Let $y=\frac{\beta}{\gamma}+\frac{\gamma}{\beta}=\frac{\beta^2+\gamma^2}{\beta\gamma}=\frac{(\beta+\gamma)^2-2\beta\gamma}{\beta\gamma}=\frac{(-\alpha)^2-2\beta\gamma}{\beta\gamma}$

$=\frac{\alpha^2}{\beta\gamma}-2$

$\Rightarrow y+2=\frac{\alpha^2}{\beta\gamma}=\frac{\alpha^3}{\alpha\beta\gamma}=\frac{\alpha^3}{-r}$

$\Rightarrow \alpha^3+r(y+2)=0$ ——— (1)

Also $\alpha^3+q\alpha+r=0$ ——— (2)

on substraction, we have

$-q\alpha+r(y+2)-r=0$

$\Rightarrow -q\alpha+r(y+2-1)=0$

$\Rightarrow -q\alpha+r(y+1)=0$

$\therefore q\alpha=r(y+1)$

$\therefore \alpha=\frac{r(y+1)}{q}$

But α is a root of the equation $x^3+qx+r=0$

$\Rightarrow \frac{r^3(y+1)^3}{q^3}+q\cdot\frac{r(y+1)}{q}+r=0$

$\Rightarrow r^3(y+1)^3+q^3r(y+1)+rq^3=0 \Rightarrow r^2(y+1)^3+q^3(y+1)+q^3=0$

It is required equation.

Ex. If α, β, γ be the roots of the equation $x^3+qx+r=0$, find the equation whose roots are $\frac{1}{\beta}+\frac{1}{\gamma}, \frac{1}{\gamma}+\frac{1}{\alpha}, \frac{1}{\alpha}+\frac{1}{\beta}$.

Sol^n Here, $\alpha+\beta+\gamma=0$

$\beta+\gamma=-\alpha$

Let $y=\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta+\gamma}{\beta\gamma}=\frac{-\alpha}{\beta\gamma}=\frac{-\alpha^2}{\alpha\beta\gamma}=\frac{-\alpha^2}{-r}$

$=\frac{\alpha^2}{r}$

$\Rightarrow ry=\alpha^2$

Also, we have $\alpha^3+q\alpha+r=0$

$\Rightarrow \alpha(\alpha^2+q)+r=0$

$\Rightarrow \alpha[ry+q]=-r$

$\Rightarrow \alpha^2(ry+q)^2=r^2$

$\Rightarrow ry(ry+q)^2=r^2$

$\Rightarrow ry(r^2y^2+2ryq+q^2)=r^2$

$\Rightarrow r^2y^3+2rqy^2+q^2y=r$

$\Rightarrow r^2y^3+2qry^2+q^2y=r$

$\Rightarrow r^2y^3+2rqy^2+q^2y-r=0$

It is required equation.

Ex. Find the equation whose roots are the ratios of the roots α, β, γ of the cubic $x^3+qx+r=0$

Sol^n Let $y=\frac{\alpha}{\beta}$ [there will be six ratios as $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}, \frac{\beta}{\gamma}, \frac{\gamma}{\beta}, \frac{\gamma}{\alpha}, \alpha/\gamma$]

$\Rightarrow \alpha = y\beta.$

Now, $f(\beta)=0$ and $f(\alpha)=0$

$\therefore f(\beta y)=0$

Now, $f(\beta) = \beta^3+q\beta+r=0$

$f(\beta y) = \beta^3y^3+q\beta y+r=0$

By cross-multiplication, we have

$$\frac{\beta^3}{rq-rqy} = \frac{\beta}{ry^3-r} = \frac{1}{qy-qy^3}$$

$$\Rightarrow \frac{\beta^3}{qr(1-y)} = \frac{\beta}{r(y^3-1)} = \frac{1}{qy(1-y^2)}$$

$$\therefore \beta = \frac{r(y^3-1)}{qy(1-y^2)} = \frac{-r(1-y^3)}{qy(1-y)(1+y)} = \frac{-r\cancel{(1-y)}(1+y+y^2)}{qy\cancel{(1-y)}(1+y)}$$

$$= \frac{-r(1+y+y^2)}{qy(1+y)}$$

$$\& \ \beta^3 = \frac{qr(1-y)}{qy(1-y^2)} = \frac{\cancel{q}r\cancel{(1-y)}}{\cancel{q}y\cancel{(1-y)}(1+y)} = \frac{r}{y(1+y)}$$

Now, $\beta^3 = (\beta)^3$

$$\frac{r}{y}\cdot\frac{1}{1+y} = \frac{-r^3(1+y+y^2)^3}{q^3y^3(1+y)^3}$$

$$\Rightarrow r^2(1+y+y^2)^3+q^3y^2(1+y)^2=0$$

It is required eq^n of sixth degree.